

Bertotti-Robinson and soliton string solutions of $D = 5$ minimal supergravity

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We report on a series of new solutions to five-dimensional minimal supergravity. Our method applies to space-times with two commuting Killing symmetries and consists in combining dimensional reduction on two-spaces of constant curvature with reduction on a two-torus. The first gives rise to various generalized Bertotti-Robinson solutions supported by electric and magnetic fluxes, which presumably describe the near-horizon regions of black holes and black rings (strings). The second provides generating techniques based on U-duality of the corresponding three-dimensional sigma model. We identify duality transformations relating the above solutions to asymptotically flat ones and obtain new globally regular dyonic solitons. Some new extremal asymptotically flat multi-center solutions are constructed too. We also show that geodesic solutions of three-dimensional sigma models passing through the same target space point generically split into disjoint classes which cannot be related by the isotropy subgroup of U-duality.

PACS numbers:

I. INTRODUCTION

As is well-known, toroidal compactification of multidimensional supergravities and superstring effective actions to three dimensions gives rise to gravity-coupled sigma models on symmetric spaces [1, 2]. Typically the target space is a coset G/H , where G is some semi-simple group combining the manifest geometric (diffeomorphism and gauge invariance) symmetries of the initial theory together with its hidden dynamical symmetries, and H is its isotropy subgroup. This construction joins classical solutions of the initial theory into duality classes related by the action of G and opens a way to various generating techniques [3–5], of which the simplest consists in generating new solutions acting by G on some seed solution. Another application is the construction of multicenter solutions [6–8].

Particularly interesting are solutions associated with geodesic subspaces of the target space, which arise when the target variables Φ^A depend on one (or several) potential functions $\sigma(x^i)$ realizing a harmonic map between the target space and the reduced three-space x^i [9]. In the one-potential case, the reduced three-metric is asymptotically Euclidean and the harmonic function associated with solutions of the black hole type goes asymptotically to a constant value which can be shifted to zero: $\sigma(\infty) = 0$. Since σ plays the role of the affine parameter on target space geodesics, such solutions can be seen as geodesics emanating from the point $X_0 = \{\Phi^A[\sigma(\infty)]\}$. Acting on their tangent vectors by the elements of the isotropy subgroup H leaving the point X_0 intact, one can pass from one black hole solution

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to another with the same asymptotics. Usually this method is applied to generate asymptotically flat (AF) solutions by the action of H transformations on a basic seed solution of the Schwarzschild or Kerr type. For instance, this has been used in $D = 5$ to generate rotating black string solutions to vacuum gravity ($G = SL(3, R)$) [10] and to minimal supergravity ($G = G_{2(2)}$) [11] from the Kerr black string.

One can also transform geodesic solutions with a given asymptotic behavior to solutions with different asymptotics, associated with target space geodesics passing through a different fixed point. For instance, in $D = 5$ black strings and black holes are both asymptotically flat, but the point at infinity X_1 on black hole geodesics is different from the point at infinity X_0 on black string geodesics. In the case of vacuum gravity one can find G -transformations, not belonging to H , which transform X_0 into X_1 , and thus black holes into black strings or vice-versa [12, 13], and this can be extended in principle to the case of minimal supergravity [4]. Also of interest are non-asymptotically flat (NAF) solutions, in particular Bertotti-Robinson (BR) solutions with AdS asymptotics, which correspond to near-horizon limits of extremal black holes or black strings. The transformation between asymptotically flat geodesic solutions and asymptotically BR solutions has been carried out for $D = 4$ Einstein-Maxwell theory ($G = SU(2, 1)$) in [14], and sketched for $D = 5$ minimal supergravity in [4, 15]. Similar transformations between geodesic solutions with different asymptotics have also been discussed in the case of Euclidean Einstein-Maxwell-dilaton-axion theory ($G = Sp(4, R)$) [16]. The existence of such $NAF \leftrightarrow AF$ transformations makes it possible to generate asymptotically flat solutions from non-asymptotically flat seeds, as will be demonstrated in the present paper.

Another, quite different question is whether *all* target space geodesics with the same harmonic potential $\sigma(x^i)$ and passing through the same fixed point X_0 can be related by the action of the isotropy subgroup H . As we shall show on the examples of $D = 5$ vacuum gravity and minimal supergravity, the answer is negative. As a consequence of the existence of a number of invariants which are preserved by H -transformations, the set of geodesics through a given fixed point, and thus by transitivity the full solution space (the set of all target space geodesics) splits into disjoint equivalence classes which cannot be transformed into each other by G -transformations.

Our considerations here will be focussed on five-dimensional minimal supergravity (MSG5), which attracted special attention in relation with black rings and black strings [11, 17]. The target space of the corresponding three-dimensional sigma model is the coset $G/H = G_{2(+2)}/((SL(2, R) \times SL(2, R)))$ [18, 19]. This theory is a particular case of the more general Einstein-Maxwell-Chern-Simons theory with arbitrary coupling and cosmological constant for which we have obtained recently [20] a number of physically interesting NAF solutions via compactification on two-dimensional constant curvature spaces. This reduction (with no Kaluza-Klein vectors), recalled in the next section, leads to a three-dimensional theory possessing non-trivial solutions of the Banados-Teitelboim-Zanelli (BTZ), self-dual and Gödel type. Their five-dimensional uplifting gives rise to generalized Bertotti-Robinson metrics which could serve as near-horizon limit of yet unknown extremal AdS black rings [21]. In Sect. 3, we briefly review the sigma model arising from toroidal reduction of MSG5, and discuss the isotropy and non-isotropy transformations between geodesic solutions. The map between the five-dimensional non-asymptotically flat BTZ black string and the Schwarzschild black string is presented in Sect. 4.

Our main new results stem from the application of the $NAF \leftrightarrow AF$ map to the non-asymptotically flat Gödel string (unrelated to the five-dimensional Gödel black holes). First, we show in Sect. 5 that, while the Gödel string cannot be G -transformed into the Schwarzschild black string, it can be transformed into the Euclidean Schwarzschild string (the product of the four-dimensional Euclidean Schwarzschild metric by the time axis). In Sect. 6, we generate from the Gödel string a non-singular (geodesically complete) locally AF metric (with spatial sections which are asymptotically $R^3 \times S^1$), which to our knowledge is the first exact solution of five-dimensional supergravity describing a non-BPS regular soliton. This soliton is supported by electric and magnetic fluxes, endowed with a NUT parameter and has zero Schwarzschild mass. A further transformation exchanging the mass and NUT parameters leads to a NUT-less soliton with positive mass, which is also regular everywhere. Another unexpected byproduct of our investigation is the existence of a signature-changing transformation between asymptotically flat solutions with Lorentzian signature and anti-Euclidean signature (five timelike coordinates) respectively.

Some of our generalized BR solutions correspond to null geodesics of the target space, leading to several classes of multicenter solutions which are presented in Sect. 7. The possibility of applying our BR-like solutions to the generation of rotating solutions is discussed in Sect. 8. Our results are summarized in the closing section.

II. REDUCTION OF MSG5 ON CONSTANT CURVATURE TWO-SPACES

The bosonic sector of MSG5 is described by the action

$$S_5 = \frac{1}{16\pi G_5} \int d^5x \left[\sqrt{|g_{(5)}|} \left(R_{(5)} - \frac{1}{4} F_{(5)}^{\mu\nu} F_{(5)\mu\nu} \right) - \frac{1}{12\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} F_{(5)\mu\nu} F_{(5)\rho\sigma} A_{(5)\lambda} \right], \quad (2.1)$$

where $F_{(5)} = dA_{(5)}$, $\mu, \nu, \dots = 1, \dots, 5$. The sign convention for the five-dimensional antisymmetric symbol will be fixed throughout this paper by assuming that $\epsilon^{12345} = +1$, with the space-time coordinates numbered according to their order of appearance in the relevant five-dimensional metric. The five-dimensional Maxwell-Chern-Simons and Einstein equations following from the action (2.1) are

$$\partial_\mu(\sqrt{|g_{(5)}|}F_{(5)}^{\mu\nu}) = \frac{1}{4\sqrt{3}}\epsilon^{\nu\rho\sigma\tau\lambda}F_{(5)\rho\sigma}F_{(5)\tau\lambda}, \quad (2.2)$$

$$R_{(5)\nu}^\mu - \frac{1}{2}R_{(5)}\delta_\nu^\mu = \frac{1}{2}F_{(5)}^{\mu\rho}F_{(5)\nu\rho} - \frac{1}{8}F_{(5)}^2\delta_\nu^\mu. \quad (2.3)$$

Let us assume for the five-dimensional metric and the vector potential the direct product ansatz

$$ds_{(5)}^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta + a^2 d\Sigma_k, \quad (2.4)$$

where $\alpha, \beta, \gamma = 1, 2, 3$, and the two-metrics for $k = \pm 1, 0$ are

$$d\Sigma_1 = d\theta^2 + \sin^2\theta d\varphi^2, \quad d\Sigma_0 = d\theta^2 + \theta^2 d\varphi^2, \quad d\Sigma_{-1} = d\theta^2 + \sinh^2\theta d\varphi^2 \quad (2.5)$$

$$f_1 = -\cos\theta, \quad f_0 = \frac{1}{2}\theta^2, \quad f_{-1} = \cosh\theta \quad (2.6)$$

with $\varphi \in [0, 2\pi]$ and $\theta \in [0, \pi]$ for $k = 1$ and $\theta \in [0, \infty]$ for $k = 0, -1$. The vector potential is decomposed as

$$A_{(5)} = A_\alpha(x^\gamma)dx^\alpha + e f_k d\varphi. \quad (2.7)$$

Here the moduli e and a^2 are taken to be constant and real (though for generality we do not assume outright a^2 to be positive).

Following [20] one can show that the Eqs. (2.3) reduce to those following from the three-dimensional theory

$$S = \frac{1}{2\kappa} \int d^3x \left[\sqrt{|g|} \left(\mathcal{R} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} - 2\lambda \right) - \frac{\mu}{4} \epsilon^{\alpha\beta\gamma} F_{\alpha\beta} A_\gamma \right] \quad (2.8)$$

with $\kappa = 2G_5/|a^2|$ and the identification of parameters:

$$\lambda = (e^2 - 4ka^2)/4a^4, \quad \mu = g/|a^2|, \quad (g = 2e/\sqrt{3}), \quad (2.9)$$

provided the three-dimensional scalar curvature is further constrain by

$$\mathcal{R} = (e^2 - 6ka^2)/2a^4. \quad (2.10)$$

The three-dimensional theory defined by the action (2.8) is Maxwell-Chern-Simons electrodynamics coupled to cosmological Einstein gravity. Several classes of exact stationary solutions to this theory with constant Ricci scalar are known [22–26].

A. BTZ solution

The first class corresponds to neutral (vacuum) three-dimensional solutions with

$$e^2 = 3ka^2, \quad \mathcal{R} = 6\lambda = -\frac{3k}{2a^2}. \quad (2.11)$$

The BTZ black hole is a vacuum solution of three-dimensional gravity with negative $\lambda = -l^{-2}$, so that, for $a^2 > 0$,

$$k = +1, \quad a^2 = \frac{l^2}{4}, \quad e^2 = \frac{3l^2}{4}. \quad (2.12)$$

Uplifting this to five dimensions according to (2.4), we obtain the following two-parameter family of solutions:

$$\begin{aligned} ds_{(5)}^2 &= -\frac{1}{2a}(r - Ma) dt^2 - J dt dz + 2a(r + Ma) dz^2 \\ &\quad + a^2 \left(\frac{dr^2}{r^2 + J^2/4 - M^2 a^2} + d\theta^2 + \sin^2\theta d\varphi^2 \right), \\ A_{(5)} &= -\sqrt{3}a \cos\theta d\varphi \end{aligned} \quad (2.13)$$

(r is related to the BTZ radial coordinate r_{BTZ} by $r_{BTZ}^2 = 2a(r + Ma)$). The local isometry group of these solutions is $SO(2, 2) \times SO(3)$.

The solution (2.13) with $z \in R$ coincides with the decoupling (near-horizon) limit of the general five-dimensional black string [11]. With z periodically identified, the solution (2.13) may be interpreted as a NAF black ring rotating along the S^1 . Moreover, it is the near-horizon limit of the asymptotically flat black ring with horizon $S^1 \times S^2$.

B. Self-dual solutions

The second class is that of the “self-dual” solutions of [27] and [28] which asymptote to the extreme ($J = Ml$) BTZ solution (2.13). For these solutions, $F^2 = 0$ (but $F_{\alpha\beta} \neq 0$), and the constant Ricci scalar has again the BTZ value $\mathcal{R} = 6\lambda \equiv -6l^{-2}$, leading to $l = 2a$, $\mu = \pm 2/a$, so that the characteristic exponent μl of [27] takes the value ± 4 . The corresponding five-dimensional solution is:

$$\begin{aligned} ds_{(5)}^2 &= \frac{1}{a} \left[-(r - aM_{\pm}(r)) dt^2 - 2aM_{\pm}(r) dt dz + (r + aM_{\pm}(r)) dz^2 \right] \\ &\quad + a^2 \left(\frac{dr^2}{r^2} + d\theta^2 + \sin^2 \theta d\varphi^2 \right), \\ A_{(5)} &= \sqrt{3} \left[c \left(\frac{r}{a} \right)^{\mp 2} (dt - dz) \mp a \cos \theta d\varphi \right], \quad M_{\mu}(r) = M - \frac{3c^2}{4 \pm 1} \left(\frac{r}{a} \right)^{\mp 4} \end{aligned} \quad (2.14)$$

with c a dimensionless parameter.

C. Gödel solution

The third class, corresponding to so-called three-dimensional Gödel black holes (no relation with the five-dimensional Gödel black holes), was given in [23] and [24] (in the case where the Chern-Simons term for gravity is absent). These solutions are closely related to the warped AdS_3 black hole solutions of topologically massive gravity [22, 24–26]. Using the notations of [24], the three-dimensional solutions, characterized by a dimensionless constant $\beta^2 = (1 - 4\lambda/\mu^2)/2$, have a constant Ricci scalar $\mathcal{R} = (1 - 4\beta^2)\mu^2/2$, so that the constraint (2.10) implies

$$\lambda = \frac{5\mu^2}{16}, \quad (2.15)$$

leading to

$$\beta^2 = \frac{k}{\mu^2 a^2} = k \frac{a^2}{g^2}. \quad (2.16)$$

Comparing (2.15) and (2.9), we see that for these solutions the constant k must be given by

$$k = -\frac{e^2}{6a^2}, \quad (2.17)$$

so that, assuming $a^2 > 0$, $k = -1$, and $\beta^2 = -1/8$. The resulting five-dimensional solution, derived in [20], may be written in the form

$$\begin{aligned} ds_{(5)}^2 &= -(dt - gy d\psi)^2 + \frac{g^2}{8} \left[\frac{dy^2}{1 - y^2} + (1 - y^2) d\psi^2 + \frac{dx^2}{x^2 - 1} + (x^2 - 1) d\varphi^2 \right] \\ A_{(5)} &= -\frac{3}{2} (dt - gy d\psi) + \frac{\sqrt{3}}{2} gx d\varphi, \end{aligned} \quad (2.18)$$

with $x^2 > 1$, $y^2 < 1$. The local isometry group of this metric is $SO(2, 1) \times SO(2) \times SO(2, 1)$. Similarly to the BTZ metric (2.13), it is geodesically complete.

Let us note that Eq. (2.17) can also be solved by $k = +1$, $a^2 = -\overline{a^2} < 0$ (reduction on a timelike two-sphere). The resulting “antiGödel” metric, with the unphysical signature $(- - - -)$, may again be written in the form (2.18), but with $g \rightarrow -g$, and $x^2 < 1$, $y^2 > 1$. Remarkably, as we shall see in Sect. 6, the Gödel and antiGödel solutions, with different spacetime signatures, can also be transformed into each other by sigma-model transformations.

III. TOROIDAL REDUCTION AND SIGMA-MODEL TRANSFORMATIONS

A. General setup

All the solutions discussed above admit three commuting Killing vectors. In this case, beside reduction on a constant curvature two-surface, one can also carry out toroidal reduction relative to any two ∂_a ($a = 1, 2$) of these three Killing vectors, according to the $GL(2, R)$ -covariant Kaluza-Klein ansatz

$$ds_{(5)}^2 = \lambda_{ab}(dx^a + a_i^a dx^i)(dx^b + a_j^b dx^j) + \tau^{-1} h_{ij} dx^i dx^j, \quad (3.1)$$

$$A_{(5)} = \sqrt{3}(\psi_a dx^a + A_i dx^i) \quad (3.2)$$

($i, j = 3, 4, 5$) where $\tau = -\det\lambda$. The Maxwell and Kaluza-Klein vector fields are then dualized to scalar potentials ν (magnetic¹) and ω_a (twist). In performing this dualization, we must take care that the scalar potential τ can be positive (for most of the solutions considered here) or negative (in the special case of the anti-Gödel solutions with (5-) signature). In this case $\sqrt{|g_{(5)}|} = \varepsilon\tau\sqrt{h}$, where $\varepsilon = \text{sign}(\tau)$, and the dualization equations of [18, 19] are modified to

$$F^{ij} = a^{aj}\partial^i\psi_a - a^{ai}\partial^j\psi_a + \varepsilon\frac{1}{\tau\sqrt{h}}\epsilon^{ijk}\eta_k, \quad \eta_k = \partial_k\nu + \epsilon^{ab}\psi_a\partial_k\psi_b \quad (3.3)$$

and

$$\lambda_{ab}G^{bij} = \varepsilon\frac{1}{\tau\sqrt{h}}\epsilon^{ijk}V_{ak}, \quad V_{ak} = \partial_k\omega_a - \psi_a(3\partial_k\nu + \epsilon^{bc}\psi_b\partial_k\psi_c), \quad (3.4)$$

with $G_{ij}^b \equiv \partial_i a_j^b - \partial_j a_i^b$. After dualization, the reduced field equations derive from the reduced action (up to a multiplicative constant)

$$S_3 = \int d^3x\sqrt{h}\left(-R + \frac{1}{2}G_{AB}\frac{\partial\Phi^A}{\partial x^i}\frac{\partial\Phi^B}{\partial x^j}h^{ij}\right), \quad (3.5)$$

where the Φ^A ($A = 1, \dots, 8$) are the eight moduli λ_{ab} , ω_a , ψ_a , and μ . The action (3.5) describes the three-dimensional gravity coupled gauged sigma model for the eight-dimensional target space with metric:

$$dS^2 \equiv G_{AB}d\Phi^A d\Phi^B = \frac{1}{2}\text{Tr}(\lambda^{-1}d\lambda\lambda^{-1}d\lambda) + \frac{1}{2}\tau^{-2}d\tau^2 - \tau^{-1}V^T\lambda^{-1}V \\ + 3(d\psi^T\lambda^{-1}d\psi - \tau^{-1}\eta^2), \quad (3.6)$$

where λ is the 2×2 matrix of elements λ_{ab} , and ψ , V the column matrices of elements ψ_a , V_a .

The target space metric (3.6) admits fourteen Killing vectors. Nine generate manifest symmetries (generalized gauge transformations). These belong to several $GL(2, R)$ multiplets: a four-component mixed tensor M_a^b generating $GL(2, R)$ linear transformations in the (x^1, x^2) plane; a two-component contravariant vector R^a generating gauge transformations of the ψ_a ; another two-component contravariant vector N^a and a scalar Q generating translations of the dualized potentials ω_a and ν . These nine Killing vectors are supplemented by five Killing vectors L_a , P^a and T generating non-trivial hidden symmetries of the target space. The algebra generated by the full set of manifest and hidden Killing vectors J_M ($M = 1, \dots, 14$) is that of the fourteen-parameter group $G_{2(+2)}$, and the target space metric (3.6) is that of the symmetric space $G_{2(+2)}/((SL(2, R) \times SL(2, R)))$. A matrix representative of this coset can be constructed [18, 19] as a symmetric 7×7 matrix $M = M(\Phi)$, given in Appendix A, such that the target space metric is given by

$$dS^2 = \frac{1}{4}\text{Tr}(M^{-1}dMM^{-1}dM). \quad (3.7)$$

This form is manifestly invariant under the global action of the coset isometry group, generating transformation of the moduli $\Phi \rightarrow \Phi'$:

$$M(\Phi) \rightarrow M(\Phi') = P^T M(\Phi) P, \quad (3.8)$$

¹ The magnetic potential μ of [18, 19] is noted here ν to avoid confusion with the Chern-Simons coupling constant.

where the operators $P \in G = G_{2(+2)}$ are generated by the 7×7 matrix representatives j_M of the Killing vectors J_M (also given in Appendix A). These transformations leave invariant the gravitating sigma model field equations

$$\nabla_i (M^{-1} \nabla^i M) = 0, \quad (3.9)$$

$$R_{(3)ij} = \frac{1}{4} \text{Tr}(M^{-1} \partial_i M M^{-1} \partial_j M), \quad (3.10)$$

where ∇_i and $R_{(3)ij}$ are the covariant derivative and Ricci tensor associated with the reduced metric h_{ij} . The G -transformations (3.8) of the moduli matrix thus belong to the classical U -duality group connecting different solutions with the same reduced three-metric h_{ij} .

B. Geodesic solutions

All the solutions given in the preceding section admit toroidal reductions such that the moduli Φ^A depend on the three-space coordinates through a single scalar function $\sigma(x)$. As shown in [9], this potential can be chosen to be harmonic,

$$\nabla^2 \sigma = 0, \quad (3.11)$$

so that the field equations reduce to

$$\frac{d}{d\sigma} \left(M^{-1} \frac{dM}{d\sigma} \right) = 0, \quad (3.12)$$

$$R_{(3)ij} = \frac{1}{4} \text{Tr} \left(M^{-1} \frac{dM}{d\sigma} \right)^2 \partial_i \sigma \partial_j \sigma. \quad (3.13)$$

The first of these equations is the geodesic equation for the target space metric (3.6) with σ the affine parameter. It is solved by

$$M = \eta e^{\mathcal{A}\sigma}, \quad (3.14)$$

where $\eta \in G/H$ and $\mathcal{A} \in \text{Lie}(G) - \text{Lie}(H)$ are constant matrices, which transform under the action (3.8) of G according to

$$\eta' = P^T \eta P, \quad \mathcal{A}' = P^{-1} \mathcal{A} P. \quad (3.15)$$

The second equation (3.13) then reduces to

$$R_{(3)ij} = \frac{1}{4} \text{Tr}(\mathcal{A}^2) \partial_i \sigma \partial_j \sigma. \quad (3.16)$$

The sign of the spatial curvature, hence the nature of the three-geometry, depends on the sign of the constant $\text{Tr}(\mathcal{A}^2)$. This trace is invariant under general G -transformations. If the target space metric has indefinite signature, which for Lorentzian solutions is the case in presence of vector charges, then geodesics are split into three disjoint classes: a timelike class ($\text{Tr}(\mathcal{A}^2) > 0$), which includes black hole solutions; a null class ($\text{Tr}(\mathcal{A}^2) = 0$), which corresponds to extremal black holes and multi-black hole solutions; and a spacelike class ($\text{Tr}(\mathcal{A}^2) < 0$), which includes wormhole solutions. In the present paper, we will be mainly concerned with solutions of the timelike class (solutions of the null class will be discussed in Sect. 7). In that case the constant $\text{Tr}(\mathcal{A}^2)$ can be fixed to

$$\text{Tr}(\mathcal{A}^2) = 4. \quad (3.17)$$

The Einstein-scalar equations (3.16) then determine the reduced metric $h_{ij}(x)$ (up to coordinate transformations):

$$ds_{(3)}^2 \equiv h_{ij} dx^i dx^j = dr^2 + (r^2 - m^2)(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (3.18)$$

and the scalar potential $\sigma(x)$ (up to linear transformations).

$$\sigma = \ln f, \quad f(x) = \frac{r - m}{r + m}. \quad (3.19)$$

C. Isotropy transformations

We can regard geodesic solutions as curves in target space passing through the point $X_0 = \{\Phi^A(x_0)\}$ where x_0 is some characteristic point in the reduced three-space. Often (but not necessarily), one takes $x_0 = \infty$, and chooses for the harmonic potential $\sigma(x)$ a gauge such that $\sigma(\infty) = 0$. In that case, the constant matrix $\eta = M(\infty)$ specifies the asymptotic nature of the solution under consideration. In this paper, we consider only (not necessarily black) string solutions. In the locally asymptotic Minkowskian (LAM) case (with possible Misner string singularities), in a gauge where $\lambda(\infty) = \text{diag}(-1, 1)$ and the other moduli vanish at infinity, the corresponding matrix η is given by

$$\eta_S = \text{diag}(-1, 1, -1, -1, 1, -1, 1). \quad (3.20)$$

This is invariant under the transformations $P \in H = SL(2, R) \times SL(2, R)$ generated by the eight elements of the isotropy subalgebra

$$h_S = \{n^0 + \ell_0, n^1 - \ell_1, m_1^0 + m_0^1, r^0 + p_0, r^1 - p_1, q - t\}. \quad (3.21)$$

Starting from a given geodesic solution, e.g. the Schwarzschild black string

$$\begin{aligned} ds_{(5)}^2 &= -f(r) dt^2 + dz^2 + f^{-1}(r) [dr^2 + (r^2 - m^2) d\Omega_2^2], \\ A_{(5)} &= 0, \end{aligned} \quad (3.22)$$

one can generate other LAM solutions through the action of isotropy transformations $P \in H_S$. The question arises, whether one can obtain *all* geodesics passing through $X(\infty)$ in this way?

Consider for instance the simple example of the coset $SL(n, R)/SO(n-2, 2)$ ($(n+2)$ -dimensional vacuum gravity). The dimensions of the invariance group G and of the isotropy subgroup H are $n_G = n^2 - 1$ and $n_H = n(n-1)/2$. The number of charges in the charge matrix \mathcal{A} (with $\text{Tr}(\mathcal{A}) = 0$) is equal to the dimension of the coset $n_c = n_G - n_H = (n+2)(n-1)/2$. However, in a given equivalence class (under isotropy transformations), the number of independent charges is lower. The reason is that such transformations preserve the $(n-1)$ invariants $\text{Tr}(\mathcal{A}^2), \dots, \text{Tr}(\mathcal{A}^n)$, so that the number of independent charges is only $n_c - (n-1) = n(n-1)/2 = n_H$. Furthermore, not all geodesic solutions with given values c_i ($i = 1, \dots, n-1$) of the $(n-1)$ trace invariants are equivalent to some given solution with the same values for these invariants. The corresponding charges belong to an $n(n-1)/2$ dimensional variety V which is the intersection of the i -dimensional varieties $\text{Tr}(\mathcal{A}^{i+1}) = c_{i+1}$ ($i = 2, \dots, n$) and may have several connected components.

Take the case of five-dimensional Lorentzian vacuum gravity (E5) reduced to three Euclidean dimensions. The target space is $SL(3, R)/SO(2, 1)$. Consider locally asymptotically flat geodesic solutions $M = \eta_S \exp[\mathcal{A}\sigma]$, with $\eta_S = \text{diag}(-1, 1, -1)$. A necessary condition for regularity of these solutions is $\det \mathcal{A} = 0$. After using the tracelessness and normalization conditions $\text{Tr}(\mathcal{A}) = 0$, $\text{Tr}(\mathcal{A}^2) = 2$, leading to $\mathcal{A}^3 = \mathcal{A}$, the charge matrix \mathcal{A} can be diagonalized to one of the possible three forms

$$\mathcal{A}_1 = \text{diag}(1, 0, -1), \quad \mathcal{A}_2 = \text{diag}(0, 1, -1), \quad \mathcal{A}_3 = \text{diag}(1, -1, 0), \quad (3.23)$$

which are inequivalent (cannot be transformed into each other by similarity transformations belonging to $H = SO(2, 1)$). The first one leads to the class of the Schwarzschild black string (3.22) (S), which is the direct product of the four-dimensional Lorentzian Schwarzschild black hole by spacelike S^1 , and other black strings, as well as black holes. The second one leads to the class of soliton strings generated from the Euclidean Schwarzschild string (ES), the direct product of the four-dimensional Euclidean Schwarzschild solution by the timelike real axis:

$$ds_{(5)}^2 = -dt^2 + f(r) dz^2 + f^{-1}(r) [dr^2 + (r^2 - m^2) d\Omega_2^2] \quad (3.24)$$

(which is also regular if the coordinate z is periodically identified with suitable period). And the third leads to the class generated from the singular solution

$$ds_{(5)}^2 = -f(r) dt^2 + f^{-1}(r) dz^2 + dr^2 + (r^2 - m^2) d\Omega_2^2. \quad (3.25)$$

So in this case geodesic solutions on a given three-dimensional reduced metric fall into three distinct equivalence classes². We will show in the following that this result holds also for minimal five-dimensional supergravity.

² The obstruction discussed here is clearly different from that considered in [29], which arises when the charge matrix \mathcal{A} has complex eigenvalues.

D. Non-isotropy transformations

Another question is whether one can transform a solution corresponding to a geodesic passing through a given point X (for instance LAM) to a solution corresponding to different asymptotics, i.e. passing through a different point X' of the target space. Such transformations $P_{XX'} \notin H$ will lead from η_X to

$$\eta_{X'} = P_{XX'}^T \eta_X P_{XX'} \neq \eta_X. \quad (3.26)$$

Three quite different type of transformations are actually concerned. The transformation $P_{XX'}$ can be a generalized gauge transformation, which does not modify the intrinsic solution. Or it can relate asymptotically flat (AF) solutions with intrinsically different asymptotics, for instance transform black strings into black holes [12, 13]. Or finally it can relate AF and non-asymptotically flat (NAF) solutions, transforming for instance an AF black hole to another exact solution which is its near-horizon limit and back [14, 16]. In all cases, since the U -duality group acts transitively on the target space, inequivalent geodesics passing through X will be transformed into inequivalent geodesics passing through X' . Thus, the existence of several distinct equivalence classes of geodesics through a given point X of target space actually means that the solution space has several disjoint components, which cannot be related by invariance group transformations, irrespective of the asymptotics involved.

IV. FROM BTZ TO SCHWARZSCHILD

We first give a non-trivial example of relating solutions with different asymptotics, namely the BTZ ring (2.13) and Schwarzschild black string (3.22). After toroidal reduction relative to ∂_t and ∂_z , the three-dimensional reduced metric h_{ij} is in both cases (3.18), with $m^2 \equiv M^2 a^2 - J^2/4$ in the BTZ case. This means that the corresponding 7×7 matrix representatives M_S (Schwarzschild) and M_B (BTZ) may be related by a $G_{2(2)}$ transformation,

$$M_B = P_{SB}^T M_S P_{SB}. \quad (4.1)$$

To construct the transformation matrix P_{SB} , we can use the fact that both M_S and M_B are geodesic solutions $M = \eta e^{\mathcal{A}\sigma}$ with $\sigma = \ln f$, so that their asymptotic and charge matrices η and \mathcal{A} are related by

$$\eta_B = P_{SB}^T \eta_S P_{SB}, \quad (4.2)$$

$$\mathcal{A}_B = P_{SB}^{-1} \mathcal{A}_S P_{SB}, \quad (4.3)$$

i.e. P_{SB} is the inverse of a similarity transformation P_{BS} bringing the matrix \mathcal{A}_B into the diagonal form \mathcal{A}_S , normalized by the constraint (4.2), and subject to additional constraints ensuring that it belongs to G_2 . Necessary conditions for the transformation matrix P_{SB} to belong to G_2 are $P_{SB} \in SO(4, 3)$, which implies

$$P_{SB}^{-1} = K P_{SB}^T K, \quad \det(P_{SB}) = +1, \quad (4.4)$$

where K is the matrix

$$K = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad (4.5)$$

but these conditions are not sufficient.

Without loss of generality, we consider only in the following the static case ($J = 0$, $M = m/a$). The general solution with $J \neq 0$ can be recovered from this by a Lorentz boost in the 2-Killing space (a trivial G_2 transformation). The reduction of (2.13) leads to the scalar potentials

$$\begin{aligned} \lambda &= \frac{1}{a} \begin{pmatrix} -r+m & 0 \\ 0 & r+m \end{pmatrix}, \quad \tau = \frac{r^2 - m^2}{a^2}, \quad \psi = 0, \\ \omega &= 0, \quad \nu = \frac{r}{a}. \end{aligned} \quad (4.6)$$

From these potentials one constructs, according to the prescriptions of [18, 19], the 7×7 matrix representative

$$M_B = \frac{m^2}{r^2 - m^2} \times \begin{pmatrix} \frac{r-m}{a} & 0 & 0 & 0 & -\frac{r(r-m)}{m^2} & 0 & 0 \\ 0 & -\frac{r+m}{a} & 0 & -\frac{r(r+m)}{m^2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{a^2}{m^2} & 0 & 0 & -\frac{r^2}{m^2} & \sqrt{2}\frac{ar}{m^2} \\ 0 & -\frac{r(r+m)}{m^2} & 0 & -\frac{a(r+m)}{m^2} & 0 & 0 & 0 \\ -\frac{r(r-m)}{m^2} & 0 & 0 & 0 & \frac{a(r-m)}{m^2} & 0 & 0 \\ 0 & 0 & -\frac{r^2}{m^2} & 0 & 0 & -\frac{m^2}{a^2} & \sqrt{2}\frac{r}{a} \\ 0 & 0 & \sqrt{2}\frac{ar}{m^2} & 0 & 0 & \sqrt{2}\frac{r}{a} & -\frac{r^2+m^2}{m^2} \end{pmatrix}. \quad (4.7)$$

The resulting constant matrices η and \mathcal{A} are

$$\eta_B = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad (4.8)$$

$$\mathcal{A}_B = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & M^{-1} & 0 & 0 \\ 0 & -1 & 0 & -M^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}M \\ 0 & -M & 0 & -1 & 0 & 0 & 0 \\ M & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}M^{-1} \\ 0 & 0 & \sqrt{2}M^{-1} & 0 & 0 & \sqrt{2}M & 0 \end{pmatrix}. \quad (4.9)$$

On the other hand, the matrix representative for the Schwarzschild black string

$$M_S = \text{diag}(-f, 1, -f^{-1}, -f^{-1}, 1, -f, 1) \quad (4.10)$$

corresponds to the constant matrices

$$\eta_S = \text{diag}(-1, 1, -1, -1, 1, -1, 1), \quad (4.11)$$

$$\mathcal{A}_S = \text{diag}(1, 0, -1, -1, 0, 1, 0). \quad (4.12)$$

The procedure outlined above for determining the transformation matrix P_{SB} does not ensure that it belongs to the group G_2 . However, educated guesses show that that this matrix can be written as the product of two elementary G_2 transformations, i.e. exponentials of g_2 generators, as given in matrix form in [18] (Appendix A) and [19]. First, the transformation

$$P_{BS0} = \exp[\alpha_0(q+t)], \quad \alpha_0 = -\pi/4, \quad (4.13)$$

acting bilinearly on M_B transforms η_B to η_S and \mathcal{A}_B to

$$\mathcal{A}'_B = \frac{1}{4M} \times \quad (4.14)$$

$$\begin{pmatrix} (M+1)^2 & 0 & 0 & 0 & -(M^2-1) & 0 & 0 \\ 0 & (M-1)^2 & 0 & (M^2-1) & 0 & 0 & 0 \\ 0 & 0 & -2(M^2+1) & 0 & 0 & 0 & \sqrt{2}(M^2-1) \\ 0 & -(M^2-1) & 0 & -(M+1)^2 & 0 & 0 & 0 \\ (M^2-1) & 0 & 0 & 0 & -(M-1)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(M^2+1) & -\sqrt{2}(M^2-1) \\ 0 & 0 & -\sqrt{2}(M^2-1) & 0 & 0 & \sqrt{2}(M^2-1) & 0 \end{pmatrix}. \quad (4.15)$$

This last charge matrix may be transformed to \mathcal{A}_S by the action of transformations generated by the isotropy subalgebra (3.21). The simplest such transformation

$$P_{BS1} = \exp[\beta(q - t)], \quad \beta = -\ln(M)/2 \quad (4.16)$$

leads to the G_2 transformation $P_{BS} = P_{BS0}P_{BS1}$ transforming the BTZ black ring into the Schwarzschild black string

$$P_{BS} = \frac{1}{2} \begin{pmatrix} \sqrt{2}M^{-1/2} & 0 & 0 & 0 & -\sqrt{2}M^{-1/2} & 0 & 0 \\ 0 & \sqrt{2}M^{-1/2} & 0 & \sqrt{2}M^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & M & 0 & 0 & M & \sqrt{2}M \\ 0 & -\sqrt{2}M^{1/2} & 0 & \sqrt{2}M^{1/2} & 0 & 0 & 0 \\ \sqrt{2}M^{1/2} & 0 & 0 & 0 & \sqrt{2}M^{1/2} & 0 & 0 \\ 0 & 0 & M^{-1} & 0 & 0 & M^{-1} & -\sqrt{2}M^{-1} \\ 0 & 0 & -\sqrt{2} & 0 & 0 & \sqrt{2} & 0 \end{pmatrix}. \quad (4.17)$$

This transformation is not unique, as it can be right-factored by any transformation generated by the element $r^1 - p_1$ of h_S , which commutes with $\mathcal{A}_S = m_0^0$.

Conversely, the BTZ black ring belongs to the continuous family of magnetostatic solutions M_α generated from the Schwarzschild black string by the transformations

$$P_\alpha = \exp[-\beta(q - t)] \exp[-\alpha(q + t)], \quad \beta = -\ln(M)/2. \quad (4.18)$$

The non-vanishing scalar potentials

$$\lambda_{00} = -\frac{r - m}{\Sigma}, \quad \lambda_{11} = \frac{r + m}{\Sigma}, \quad \nu = -\frac{r \sin(2\alpha) + \frac{a^2 - m^2}{2a} \cos(2\alpha)}{\Sigma}, \quad (4.19)$$

where

$$\Sigma = r \cos(2\alpha) - \frac{a^2 - m^2}{2a} \sin(2\alpha) + \frac{a^2 + m^2}{2a}, \quad (4.20)$$

lead to the five-dimensional solution

$$\begin{aligned} ds_{(5)}^2 &= -\frac{r - m}{\Sigma} dt^2 + \frac{r + m}{\Sigma} dz^2 + \Sigma^2 \left(\frac{dr^2}{r^2 - m^2} + d\theta^2 + \sin^2 \theta d\varphi^2 \right), \\ A_{(5)} &= \sqrt{3} \left(\frac{a^2 + m^2}{2a} \sin(2\alpha) - \frac{a^2 - m^2}{2a} \right) \cos \theta d\varphi. \end{aligned} \quad (4.21)$$

This general solution, which includes the Schwarzschild black string ($\cos(2\alpha) = 1, a = m$) and the non-rotating BTZ black ring ($\sin(2\alpha) = -1$) as special cases, can be shown to be an uplift of the magnetic Reissner-Nordström solution of four-dimensional Einstein-Maxwell theory (EM4). Any solution of EM4 can be lifted to a solution of MSG5 given by

$$\begin{aligned} ds_{(5)}^2 &= ds_{(4)}^2 + (dz + C_\mu dx^\mu)^2, \\ A_{(5)} &= \sqrt{3} A_{(4)}, \quad dC = \star dA_{(4)}. \end{aligned} \quad (4.22)$$

After reduction of (4.22) to three dimensions, only four (e.g. λ_{00} , ω_0 , ψ_0 and ν) of the eight scalar potentials are independent, the other four being related to these by the constraints

$$\lambda_{11} = 1, \quad \psi_1 = 0, \quad \lambda_{01} = \nu, \quad \omega_1 = -\psi_0. \quad (4.23)$$

The solution (4.21) does not satisfy these constraints as written. However these constraints are satisfied in the transformed coordinate system $(x^0, x^1) = (\tau, \psi)$, with

$$t = \chi [\sin(\gamma - \alpha) \psi - \cos(\gamma - \alpha) \tau], \quad (4.24)$$

$$z = \chi [\cos(\gamma + \alpha) \psi - \sin(\gamma + \alpha) \tau], \quad (4.25)$$

where

$$\tan \gamma = \frac{a - m}{a + m}, \quad \chi^2 = \frac{1}{\cos(2\gamma)} = \frac{a^2 + m^2}{2am}. \quad (4.26)$$

After this coordinate transformation, the solution (4.21) can thus be reduced to the four-dimensional magnetic Reissner-Nordström solution

$$\begin{aligned} ds_{(4)}^2 &= -\frac{r^2 - m^2}{\Sigma^2} d\tau^2 + \frac{\Sigma^2}{r^2 - m^2} dr^2 + \Sigma^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \\ A_{(4)} &= \left(\frac{a^2 + m^2}{2a} \sin(2\alpha) - \frac{a^2 - m^2}{2a} \right) \cos \theta d\varphi, \end{aligned} \quad (4.27)$$

parameterized in a way which includes the magnetic Bertotti-Robinson solution.

V. FROM GÖDEL TO EUCLIDEAN SCHWARZSCHILD

The Gödel solution for minimal supergravity is given by (2.18). To present it in matrix form, it is convenient to introduce the dimensionless constant $b = 2m/g = \sqrt{3}m/e$, and relabel the coordinates $t \rightarrow 2t$, $y \rightarrow \cos \theta$, $\varphi \rightarrow b^2 z/m$, $\psi \rightarrow \varphi$, leading to³

$$\begin{aligned} ds_{(5)}^2 &= -4 \left[dt - \frac{m}{b} \cos \theta d\varphi \right]^2 + \frac{b^2(x^2 - 1)}{2} dz^2 + \frac{m^2}{2b^2} \left(\frac{dx^2}{x^2 - 1} + d\theta^2 + \sin^2 \theta d\varphi^2 \right), \\ A_{(5)} &= 3 \left[dt - \frac{m}{b} \cos \theta d\varphi \right] + \sqrt{3} b x dz. \end{aligned} \quad (5.1)$$

This may be toroidally reduced to three dimensions according to the ansatz (3.1), leading to the three-dimensional reduced metric (3.18) and to the metric fields

$$\lambda = \text{diag} \left(-4, \frac{b^2(x^2 - 1)}{2} \right), \quad a_\varphi = \left(-\frac{m}{b} \cos \theta, 0 \right), \quad \tau = 2b^2(x^2 - 1), \quad (5.2)$$

and the electromagnetic fields

$$\psi = \left(\sqrt{3}, bx \right), \quad A_\varphi = -\frac{\sqrt{3}m}{b} \cos \theta, \quad (5.3)$$

from which one derives the dualized potentials

$$\omega = \left(4bx, 2\sqrt{3}b^2x^2 \right), \quad \nu = \sqrt{3}bx \quad (5.4)$$

(up to irrelevant integration constants). The computation of the matrix elements leads to a coset representative $M_G(x)$ of the form (3.14), with $\sigma = \ln f(x)$, and

$$\eta_G = \frac{1}{4} \begin{pmatrix} -3 & 0 & 0 & -1 & 0 & 0 & -\sqrt{6} \\ 0 & 0 & 2\sqrt{3} & 0 & 2 & 0 & 0 \\ 0 & 2\sqrt{3} & 0 & 0 & 0 & -2 & 0 \\ -1 & 0 & 0 & -3 & 0 & 0 & \sqrt{6} \\ 0 & 2 & 0 & 0 & 0 & 2\sqrt{3} & 0 \\ 0 & 0 & -2 & 0 & 2\sqrt{3} & 0 & 0 \\ -\sqrt{6} & 0 & 0 & \sqrt{6} & 0 & 0 & 2 \end{pmatrix}, \quad (5.5)$$

$$A_G = \frac{1}{2} \begin{pmatrix} 0 & 0 & b^{-1} & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}b^{-1} \\ b & 0 & 0 & -b & 0 & 0 & 0 \\ 0 & 0 & -b^{-1} & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}b \\ b^{-1} & 0 & 0 & -b^{-1} & 0 & 0 & 0 \\ 0 & -\sqrt{2}b & 0 & 0 & -\sqrt{2}b^{-1} & 0 & 0 \end{pmatrix}. \quad (5.6)$$

³ Note that in passing from (2.18) to (5.1) we have changed the parity of the order of appearance of the five-dimensional coordinates, and so to conform with our convention for the antisymmetric symbol have changed a sign in $A_{(5)}$.

We show in Appendix B that, although the three-dimensional reduced metric is the same, this solution cannot be G_2 -transformed to the Schwarzschild black string (3.22). This means that the target space $G_{2(+2)}/((SL(2, R) \times SL(2, R)))$ admits at least two disjoint components, a black string sector generated from the Schwarzschild black string, and also containing the magnetic Bertotti-Robinson solution (2.13), as well as black hole solutions [12]; and a second component containing the 3-Gödel solution. We now show that this second component is the one generated from the Euclidean Schwarzschild string (3.24)⁴. Presumably the five-dimensional Gödel black holes of [30] would belong to the first component, however this remains to be checked.

It is actually very easy to generate from the non-asymptotically flat Gödel solution an asymptotically flat solution. $M_G(x)$ leads to a non-asymptotically flat metric because $\eta_G = M_{G33}(\infty) = -\tau^{-1}(\infty) = 0$. A generic G_2 transformation will lead to $M'_{33}(\infty) \neq 0$ (asymptotically flat metric) and with some luck negative (Lorentzian metric).

An example is the transformation

$$P_0 = \exp \left[\frac{\pi}{2} (\ell_0 + n^0) \right] = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.7)$$

Acting on η_G , this leads to the asymptotic matrix

$$\eta'_0 = P_0^T \eta_G P_0 = \frac{1}{4} \begin{pmatrix} 0 & -2\sqrt{3} & 0 & -2 & 0 & 0 & 0 \\ -2\sqrt{3} & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & -1 & -\sqrt{6} \\ -2 & 0 & 0 & 0 & -2\sqrt{3} & 0 & 0 \\ 0 & 2 & 0 & -2\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -3 & \sqrt{6} \\ 0 & 0 & -\sqrt{6} & 0 & 0 & \sqrt{6} & 2 \end{pmatrix}, \quad (5.8)$$

corresponding to an asymptotically Lorentzian (up to a coordinate transformation) $\lambda'_{ab}(\infty) dx^a dx^b = -(4/\sqrt{3}) dx^0 dx^1$ (and thus to an asymptotically Lorentzian five-metric), with $\omega'_a(\infty) = 0$, $\psi'_a(\infty) = 0$, $\nu'(\infty) = -1/\sqrt{3}$. A gauge transformation Q (linear transformation in (dx^a, dx^b) together with a translation of ν) will then transform η'_0 to the vacuum Lorentzian form

$$\eta' = Q^T \eta'_0 Q = \eta_S. \quad (5.9)$$

The corresponding full coset matrix will be

$$M'(r) = \eta_S e^{A'_0 \sigma(r)}, \quad (5.10)$$

with

$$A'_0 = Q^{-1} P_0^{-1} A_G P_0 Q. \quad (5.11)$$

⁴ The two solutions of five-dimensional vacuum gravity (3.22) and (3.24) are related by analytic continuation $t \rightarrow iz$, $z \rightarrow -it$, so that the two sectors of $SL(3, R)/SO(2, 1)$ generated from these will be related by the same analytic continuation. But this cannot be extended to the case of minimal supergravity $G_{2(+2)}/((SL(2, R) \times SL(2, R)))$, because such an analytic continuation would lead to imaginary electric potentials ψ_a .

The computation gives ($\beta = 3^{1/4}$)

$$P_0 Q = \begin{pmatrix} 0 & 0 & 2\beta^{-2} & 0 & 0 & \frac{\beta^{-2}}{2} & -\sqrt{2}\beta^{-2} \\ \frac{\beta}{2} & \frac{\beta}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\beta}{2} & \frac{\beta}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\beta^2}{2} & 0 \\ -\frac{\beta^{-1}}{2} & \frac{\beta^{-1}}{2} & 0 & \beta^{-1} & \beta^{-1} & 0 & 0 \\ -\frac{\beta^{-1}}{2} & -\frac{\beta^{-1}}{2} & 0 & -\beta^{-1} & \beta^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}, \quad (5.12)$$

$$\begin{aligned} A'_0 &= \beta^{-3}[b(n^0 - \ell_0) + b(n^1 + \ell_1) - \gamma(p_0 - r^0) - \delta(p_1 + r^1)] \\ &(\gamma = (\sqrt{3}b^{-1} - b)/2, \delta = (\sqrt{3}b^{-1} + b)/2). \end{aligned} \quad (5.13)$$

This special charge matrix includes a NUT charge, proportional to the coefficient $-b$, a Kaluza-Klein magnetic charge, proportional to b , and two electric charges (the fluxes of F_{0r} and F_{1r}), proportional to $-\gamma$ and $-\delta$.

This may be diagonalized to

$$A'_1 = P_1^{-1} A'_0 P_1 = \text{diag}(0, 1, -1, 0, -1, 1, 0) \quad (5.14)$$

through the action of transformations generated by the isotropy subalgebra (3.21):

$$P_1 = e^{\alpha_1(m_1^0 + m_0^1)} e^{(\pi/4)(p_1 - r^1)} e^{-(\pi/4)(\ell_0 + n^0)} e^{-2\alpha_2(m_1^0 + m_0^1)} \quad (5.15)$$

with $e^{\alpha_1} = \beta^{-1}b^{-1}$, $e^{\alpha_2} = \beta^{-1}$. Putting everything together, we have transformed $M_G(r)$ by the transformation

$$P = P_0 Q P_1 \quad (5.16)$$

to the diagonal form

$$M_{ES} = \text{diag}(-1, f, -f^{-1}, -1, f^{-1}, -f, 1), \quad (5.17)$$

corresponding to the Euclidean Schwarzschild string (3.24) with $A_{(5)} = 0$.

VI. GENERATING AF SOLITON SOLUTIONS

A. A continuous family of NUTty soliton solutions

A generic G_2 transformation acting on $M_G(x)$ can lead either to $\tau^{-1}(\infty) > 0$, corresponding to a five-dimensional metric with signature $(-++++)$ (asymptotically flat soliton strings, as in the preceding section), or to $\tau^{-1}(\infty) < 0$, which could correspond to a five-dimensional metric with either the signature $(-----)$ (asymptotically flat five-dimensional anti-instantons), or the signature $(---++)$. A continuous family containing both soliton strings and anti-instantons can be generated from $M_G(x)$ by the $SL(3, R)$ transformation

$$P_\alpha = \exp[\alpha(\ell_1 + n^1)] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & s & 0 & 0 & 0 & 0 \\ 0 & -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (6.1)$$

with $s \equiv \sin \alpha$, $c \equiv \cos \alpha$. This leads to the transformed scalar potentials

$$\begin{aligned}
\tau' &= -\frac{2(x^2 - 1)}{\sqrt{3}s_2 x^2 - c_2 \sigma_+ + \sigma_-}, \\
\nu' &= -\frac{(sb + \sqrt{3}cb^{-1})x}{\sqrt{3}s_2 x^2 - c_2 \sigma_+ + \sigma_-}, \\
\psi' &= \begin{pmatrix} -\frac{s_2 x^2 + \sqrt{3}(c_2 \sigma_+ - \sigma_-)}{\sqrt{3}s_2 x^2 - c_2 \sigma_+ + \sigma_-} \\ -\frac{(cb^{-1} - \sqrt{3}sb)x}{\sqrt{3}s_2 x^2 - c_2 \sigma_+ + \sigma_-} \end{pmatrix}, \\
\omega' &= \begin{pmatrix} \frac{4cb^{-1}(2s^2b^2x^2 + c_2 \sigma_+ - \sigma_-)x}{(\sqrt{3}s_2 x^2 - c_2 \sigma_+ + \sigma_-)^2} \\ \frac{-3c_2 s_2 x^4 - s_2 x^2 + 2\sqrt{3}c^2(2c_2 \sigma_+ - b^2)x^2 + s_2 \sigma_+(c_2 \sigma_+ - \sigma_-)}{(\sqrt{3}s_2 x^2 - c_2 \sigma_+ + \sigma_-)^2} \end{pmatrix}, \\
\lambda'_{00} &= -\frac{4[s_2^2 x^4 + (c_2 \sigma_+ - \sigma_-)^2]}{(\sqrt{3}s_2 x^2 - c_2 \sigma_+ + \sigma_-)^2}, \\
\lambda'_{01} &= -\frac{4sb(2c^2b^{-2}x^2 - c_2 \sigma_+ + \sigma_-)x}{(\sqrt{3}s_2 x^2 - c_2 \sigma_+ + \sigma_-)^2}, \\
\lambda'_{11} &= -\frac{3\sqrt{3}s_2 x^2(x^2 - 1) + (9s^2b^2 - c^2b^{-2})x^2 + c_2 \sigma_+ - \sigma_-}{2(\sqrt{3}s_2 x^2 - c_2 \sigma_+ + \sigma_-)^2},
\end{aligned} \tag{6.2}$$

where

$$s_2 \equiv \sin 2\alpha, \quad c_2 \equiv \cos 2\alpha, \quad \sigma_{\pm} \equiv \frac{b^2 \pm b^{-2}}{2}.$$

From the expression of τ' , one sees that the corresponding solution is a soliton string for $s_2 < 0$, and an anti-instanton for $s_2 > 0$. The non-asymptotically flat divides between the two ($s_2 = 0$) correspond to the Gödel string (2.18) for $\sin \alpha = 0$ (with both signs of $\cos \alpha$ possible), and to the anti-Gödel solution for $\cos \alpha = 0$, see below.

Inverse dualization, carried out according to (3.3)-(3.4), with $\varepsilon = -\text{sign}(s_2)$ leads to

$$\begin{aligned}
a'_{\varphi} &= \varepsilon m \begin{pmatrix} -cb^{-1} \cos \theta \\ 0 \end{pmatrix}, \\
A'_{\varphi} &= \varepsilon bm \frac{s_2(cb^{-2} + \sqrt{3}s)x^2 - (s - \sqrt{3}cb^{-2})(c_2 \sigma_+ - \sigma_-)}{\sqrt{3}s_2 x^2 - c_2 \sigma_+ + \sigma_-} \cos \theta.
\end{aligned} \tag{6.3}$$

The resulting five-dimensional metric and gauge can be put in a simple form by defining the real parameter $\beta^2 = -\sqrt{3}s_2/2$, and making the coordinate redefinitions

$$\begin{aligned}
t &\rightarrow \frac{\sqrt{3}}{2} t, \quad x \rightarrow \frac{r}{\mu}, \quad z \rightarrow \frac{2}{\sqrt{3}} \beta z \quad (\beta^2 > 0), \\
t &\rightarrow \frac{\sqrt{3}}{2} t, \quad r \rightarrow i \frac{r}{\mu}, \quad z \rightarrow -\frac{2}{\sqrt{3}} i \beta z \quad (\beta^2 < 0),
\end{aligned} \tag{6.4}$$

leading to

$$\begin{aligned}
ds'^2_{(5)} &= -\frac{r^4 + 3\nu^4}{(r^2 - \nu^2)^2} \left[dt + 2\varepsilon N \cos \theta d\varphi + \frac{2(Nr^2 + P\nu^2)r}{r^4 + 3\nu^4} dz \right]^2 \\
&\quad + \varepsilon(r^2 - \nu^2) \left[\frac{r^2 - \mu^2}{r^4 + 3\nu^4} dz^2 + \frac{dr^2}{r^2 - \mu^2} + d\theta^2 + \sin^2 \theta d\varphi^2 \right], \\
A'_{(5)} &= -\frac{\sqrt{3}}{2} \left[\frac{r^2 + 3\nu^2}{r^2 - \nu^2} (dt + 2\varepsilon N \cos \theta d\varphi) + \frac{2(N + P)r}{r^2 - \nu^2} dz - 2\varepsilon P \cos \theta d\varphi \right].
\end{aligned} \tag{6.5}$$

This solution, with signature $(-+++)$ for $\varepsilon > 0$, or $(----)$ for $\varepsilon < 0$, depends on two real parameters N (NUT charge) and P (the magnetic charge is $N - P$), with

$$\begin{aligned} N &= -\frac{mc}{\sqrt{3}b}, \quad P = bms, \\ \mu^2 &= m^2|\beta^2| = 3\varepsilon NP, \\ \nu^2 &= -\varepsilon m^2 \frac{(c_2\sigma_+ - \sigma_-)}{2} = \varepsilon \frac{P^2 - 3N^2}{2}. \end{aligned} \quad (6.6)$$

For the exceptional value $\alpha = \varepsilon'\pi/2$ ($c = 0$, $s = \varepsilon'$, $s_2 = 0$, $c_2 = -1$, $-c_2\sigma_+ + \sigma_- = b^2$), τ' is negative in the sector $x^2 > 1$ and, after inverse dualisation according to (3.3)-(3.4) and time rescaling $t \rightarrow t/2$, the five-dimensional solution reduces to

$$\begin{aligned} ds'^2_{(5)} &= -(\mathrm{d}t - g'x \mathrm{d}\psi)^2 - \frac{g'^2}{8} \left[\frac{\mathrm{d}x^2}{x^2 - 1} + (x^2 - 1)\mathrm{d}\psi^2 + \frac{\mathrm{d}y^2}{1 - y^2} + (1 - y^2)\mathrm{d}\varphi^2 \right] \\ A'_{(5)} &= \frac{3}{2}(\mathrm{d}t - g'x \mathrm{d}\psi) + \frac{\sqrt{3}}{2}g'y \mathrm{d}\varphi, \end{aligned} \quad (6.7)$$

where we have put $y = \cos\theta$, $\psi = z/b^2m$, and $g' = -2\varepsilon'bm$. This is recognized as the anti-Gödel solution in the symmetric form (2.18) with $g \rightarrow g'$ and the coordinate relabellings $x \leftrightarrow y$.

The metric (6.5) has a bolt at $r^2 = \mu^2$, where it is regular (if $\nu^2 \neq \mu^2$) provided the coordinate z is periodically identified with period $\pi\sqrt{3}(P^2 + 3N^2)/\mu$, and is singular at $r^2 = \nu^2$, unless $\nu^2 < 0$. It follows that (with this periodic identification of the coordinate z) this two-parameter solution is for $\varepsilon = +1$ a regular soliton ring provided the ratio of the two parameters lies in the range

$$0 < \frac{P}{N} < 3 + 2\sqrt{3}. \quad (6.8)$$

The bolt at $r^2 = \mu^2$ is extreme for $\mu^2 = 0$. The near-extreme, near-bolt regime corresponds to

$$\mu = |\beta|m, \quad r = |\beta|\bar{r} \quad (\beta \rightarrow 0). \quad (6.9)$$

This can be achieved in two ways. Either N is held fixed, and P goes to zero ($\nu^2 = -\varepsilon 3N^2/2$) with $|\beta^2|$, leading (up to coordinate rescalings) to the Gödel solution in its original form (5.1). Or P is held fixed, and N goes to zero ($\nu^2 = \varepsilon P^2/2$) with $|\beta^2|$, leading (again up to coordinate rescalings) to the anti-Gödel solution (6.7).

B. A class of NUTless, massive solitons

The asymptotically locally flat solution (6.5) presents two defects: 1) it is massless; 2) its NUT singularity prevents it from being truly asymptotically flat. Both defects can be cured by acting on this solution with the NUT-to-mass transformation. This $SL(2, R)$ transformation, which transforms the massless Schwarzschild-NUT solution of vacuum gravity into the Schwarzschild solution, is

$$P_{MN} = \exp[(\pi/4)(n_0 + \ell^0)]. \quad (6.10)$$

The action of this transformation on the charge matrix associated with the solution (6.5),

$$\mathcal{A}' = -\frac{1}{2\mu} \begin{pmatrix} 0 & 2N & 2N & 0 & P - N & N + P & 0 \\ -2N & 0 & 0 & N - P & 0 & 0 & -\sqrt{2}(N + P) \\ 2N & 0 & 0 & -(N + P) & 0 & 0 & \sqrt{2}(N - P) \\ 0 & P - N & -(N + P) & 0 & 2N & -2N & 0 \\ N - P & 0 & 0 & -2N & 0 & 0 & -\sqrt{2}(N + P) \\ N + P & 0 & 0 & -2N & 0 & 0 & \sqrt{2}(P - N) \\ 0 & -\sqrt{2}(N + P) & \sqrt{2}(P - N) & 0 & -\sqrt{2}(N + P) & \sqrt{2}(N - P) & 0 \end{pmatrix}. \quad (6.11)$$

leads to the transformed charge matrix $\mathcal{A}'' = P_{MN}^{-1} A' P_{MN}$:

$$\mathcal{A}'' = -\frac{1}{2\mu} \begin{pmatrix} -2N & \sqrt{2}N & 0 & 0 & \frac{P-N}{\sqrt{2}} & N+P & P-N \\ -\sqrt{2}N & 0 & -\sqrt{2}N & \frac{N-P}{\sqrt{2}} & 0 & \frac{N-P}{\sqrt{2}} & -\sqrt{2}(N+P) \\ 0 & \sqrt{2}N & 2N & -(N+P) & \frac{P-N}{\sqrt{2}} & 0 & N-P \\ 0 & \frac{P-N}{\sqrt{2}} & -(N+P) & 2N & \sqrt{2}N & 0 & N-P \\ \frac{N-P}{\sqrt{2}} & 0 & \frac{N-P}{\sqrt{2}} & -\sqrt{2}N & 0 & -\sqrt{2}N & -\sqrt{2}(N+P) \\ N+P & \frac{P-N}{\sqrt{2}} & 0 & 0 & \sqrt{2}N & -2N & P-N \\ N-P & -\sqrt{2}(N+P) & P-N & P-N & -\sqrt{2}(N+P) & N-P & 0 \end{pmatrix} \quad (6.12)$$

The resulting coset representative $M'' = \eta_s e^{\mathcal{A}'' \sigma}$ leads to the solution

$$\begin{aligned} ds_5''^2 &= \lambda_{00} \left[dt + \frac{\lambda_{01}}{\lambda_{00}} (dz - \sqrt{2}N \cos \theta d\varphi) \right]^2 - \frac{\tau}{\lambda_{00}} (dz - \sqrt{2}N \cos \theta d\varphi)^2 + \frac{dr^2}{\tau} + \frac{r^2 - \mu^2}{\tau} (d\theta^2 + \sin^2 \theta d\varphi^2), \\ A_5'' &= \sqrt{3} \left[\frac{(N-P)r + 3N^2 - P^2}{\sqrt{2}(r-\alpha)(r-\beta)} dt - \frac{N+P}{r-\beta} (dz - \sqrt{2}N \cos \theta d\varphi) - \frac{N-P}{\sqrt{2}} \cos \theta d\varphi \right], \end{aligned} \quad (6.13)$$

with

$$\begin{aligned} \tau &= \frac{r^2 - \mu^2}{(r-\alpha)(r-\beta)}, \quad \lambda_{11} = \frac{(r-\alpha)(r-2\beta+\alpha)}{(r-\beta)^2}, \\ \lambda_{01} &= \frac{(3N^2 + 4NP - 3P^2 - 4Nr)}{2\sqrt{2}(r-\beta)^2}, \quad \lambda_{00} = \frac{\lambda_{01}^2 - \tau}{\lambda_{11}}, \end{aligned} \quad (6.14)$$

where

$$\alpha = -\frac{3N+P}{2}, \quad \beta = \frac{-N+P}{2}, \quad \mu^2 = 3NP. \quad (6.15)$$

Assuming $\mu^2 > 0$, the metric (6.13) has the Minkowskian signature (τ is positive) for $r > \mu$ if

$$\mu - \alpha = \frac{(\mu + 3N)^2}{6N} > 0, \quad \mu - \beta = \frac{3N^2 + 6\mu N - \mu^2}{6N} > 0. \quad (6.16)$$

The first inequality is ensured if $N > 0$, the second is then ensured if

$$\frac{N}{\mu} > \frac{2}{\sqrt{3}} - 1 = 0.154701. \quad (6.17)$$

Near the bolt $r = \mu$, $\lambda_{00} \simeq -\lambda_{01}^2/\lambda_{11}$. One can check that $\lambda_{11}(\mu)$ is negative definite, implying $\lambda_{00}(\mu)$ negative definite. One can also check that λ_{00} is finite at the zero $r = 2\beta - \alpha$ of λ_{11} (and thus is negative definite in the range $r \geq \mu$), implying that $\lambda_{01}^2 - \tau$ can be factored by $r - 2\beta + \alpha$, so that the expression of λ_{00} can be simplified, but is still somewhat cumbersome.

For the absence of conical singularity, the coordinate z must be periodically identified with period

$$T = (-\lambda_{00}(\mu))^{1/2} \frac{(\mu - \alpha)(\mu - \beta)}{\mu} 2\pi = \sqrt{\frac{2}{3}} \frac{(3N + P + 2\mu)(3N + 3P - 2\mu)}{4\mu} 2\pi. \quad (6.18)$$

The Misner string singularity is absent if this period is equal to $\sqrt{2}N \times 4\pi$, leading to the quartic equation

$$27N^4 - 24\sqrt{3}N^3\mu + 4N\mu^3 + \mu^4 = 0. \quad (6.19)$$

This equation has the two real solutions

$$N = 0.460230\mu, \quad N = 1.45795\mu, \quad (6.20)$$

both satisfying the bound (6.17). For these values of the ratio N/μ , the solution $(ds_5''^2, A_5'')$ is a soliton ring with mass N .

VII. MULTICENTER SOLUTIONS

Null-geodesic solutions are of the form (3.14), with the charge matrix \mathcal{A} constrained by the charge balance condition [6, 7]

$$\text{Tr}(\mathcal{A}^2) = 0. \quad (7.1)$$

Null geodesics lead to a Ricci-flat, hence flat, reduced 3-space of metric h_{ij} [6, 7]. In that case, the Laplacian ∇_h^2 becomes a linear operator, so that an arbitrary number of harmonic functions may be superposed, leading to a multicenter solution

$$\sigma(\vec{x}) = \epsilon + \sum_i \frac{a_i}{|\vec{x} - \vec{x}_i|}. \quad (7.2)$$

It is easy to promote the special solutions presented in Sect. 2 to multicenter (null geodesic) solutions, provided that, after toroidal reduction relative to ∂_t and ∂_z , the reduced metric is flat. This is the case for the self-dual solution (2.14), as well as its $c = 0$ limit, the extreme BTZ solution (2.13) with $J^2 = 4M^2 a^2$, and for the Gödel solution (5.1) in the extreme case $m^2 = 0$ with $b/m = 2/g$ fixed.

A. Self-dual solutions

We first consider the self-dual solution (2.14) which contains for $c = 0$ the extreme BTZ solution. This solution can be generalized by replacing the harmonic function a/r by an arbitrary harmonic function $\sigma(\vec{x})$,

$$\begin{aligned} ds_{(5)}^2 &= \sigma^{-1} du dv + \left(M - \frac{3c^2}{4 \pm 1} \sigma^{\pm 4} \right) du^2 + \sigma^2 d\vec{x}^2, \\ A_{(5)} &= \sqrt{3} [c \sigma^{\pm 2} du \pm A_3] \quad (\nabla \wedge A_3 = \nabla \sigma), \end{aligned} \quad (7.3)$$

with $u = z - t$, $v = z + t$ ⁵. The linear superposition (7.2) leads to multicenter solutions of MSG5, which are asymptotic to the one-center solution (2.14) for $\epsilon = 0$, and asymptotically Minkowskian (up to a gauge transformation) for $\epsilon = 1$. As shown in [20], the one-center asymptotically Minkowskian solution (7.3) is an extreme black string for the lower sign, while the spacetime is geodesically complete for the upper sign.

The scalar potentials associated with (2.14) are (with $x = r/a$)

$$\lambda = \begin{pmatrix} -x + M_{\pm}(x) & -M_{\pm}(x) \\ -M_{\pm}(x) & x + M_{\pm}(x) \end{pmatrix}, \quad \tau = x^2, \quad \psi = c x^{\mp 2} (1, -1), \quad (7.4)$$

with

$$M_{\pm}(x) = M - \frac{3c^2}{4 \pm 1} x^{\mp 4}, \quad (7.5)$$

and

$$\omega = \begin{vmatrix} -3c x^{-1} (1, -1) \\ -c x^3 (1, -1) \end{vmatrix}, \quad \nu = \pm x. \quad (7.6)$$

The character of the null-geodesic solutions depends crucially on the choice of the sign \pm . For the lower sign, the representative matrix, where we have replaced x by σ^{-1} (assuming $\epsilon = 0$ in (7.2)) is

$$M = \begin{pmatrix} 0 & 0 & 0 & -M\sigma & 1 - M\sigma & 0 & 0 \\ 0 & 0 & 0 & 1 + M\sigma & M\sigma & 0 & 0 \\ 0 & 0 & -\sigma^2 & c & c & -1 & -\sqrt{2}\sigma \\ -M\sigma & 1 + M\sigma & c & -\sigma - M\sigma^2 & -M\sigma^2 & 0 & 0 \\ 1 - M\sigma & M\sigma & c & -M\sigma^2 & \sigma - M\sigma^2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2}\sigma & 0 & 0 & 0 & -1 \end{pmatrix}. \quad (7.7)$$

⁵ Again, we have changed a sign in $A_{(5)}$ because our coordinate transformation implies $\epsilon_{uv} = -\epsilon_{tz}$.

The charge matrix

$$\mathcal{A} = \begin{pmatrix} -M & M & 0 & 0 & 1 & 0 & 0 \\ -M & M & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M & M & 0 & 0 \\ 0 & 0 & 0 & -M & -M & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.8)$$

does not depend on the parameter c (which enters only the asymptotic matrix η), and is such that

$$\mathcal{A}^3 = 0, \quad \mathcal{A}^2 \neq 0. \quad (7.9)$$

The solution is presumably a G_2 transform of the vacuum (anti-)self-dual solution given in [6], with equal Kaluza-Klein electric and magnetic charges and a nilpotent charge matrix obeying (7.9).

For the upper sign, the representative matrix is of the form (3.14), with

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad (7.10)$$

$$\mathcal{A} = \begin{pmatrix} -M & M & 0 & 0 & -1 & 0 & 0 \\ -M & M & 0 & 1 & 0 & 0 & 0 \\ 12c & -12c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M & M & -12c & 0 \\ 0 & 0 & 0 & -M & -M & 12c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (7.11)$$

For $c \neq 0$, this charge matrix is nilpotent of rank six, i.e.

$$\mathcal{A}^7 = 0, \quad \mathcal{A}^6 \neq 0. \quad (7.12)$$

The corresponding geodesically complete, asymptotically $AdS_3 \times S^2$, multicenter solution has no vacuum counterpart. In the notations of [31], it belongs to the orbit \mathcal{O}_5 of $G_{2(2)}$, which also contains the supersymmetric Gödel black hole [32].

B. Gödel solutions

Trading the radial coordinate x of the Gödel solution (5.1) for $r = mx$, taking the limit $m \rightarrow 0$ with $g = 2m/b$ fixed, and replacing the harmonic function g/r by an arbitrary harmonic function $\sigma(\vec{x})$ leads to the solution

$$\begin{aligned} ds_{(5)}^2 &= -(2dt - A_3)^2 + 2\sigma^{-2} dz^2 + \frac{\sigma^2}{8} d\vec{x}^2, \\ A_{(5)} &= \frac{3}{2}(2dt - A_3) + 2\sigma^{-1} dz. \end{aligned} \quad (7.13)$$

The corresponding scalar potentials are

$$\begin{aligned} \lambda &= \begin{pmatrix} -4 & 0 \\ 0 & 2\sigma^{-2} \end{pmatrix}, \quad \omega = 8 \begin{pmatrix} \sigma^{-1} \\ \sqrt{3}\sigma^{-2} \end{pmatrix}, \\ \psi &= \begin{pmatrix} \sqrt{3} \\ 2\sigma^{-1} \end{pmatrix}, \quad \nu = 2\sqrt{3}\sigma^{-1}. \end{aligned} \quad (7.14)$$

The representative matrix

$$\begin{pmatrix} -3 & 0 & 2\sigma & -1 & -2\sqrt{3}\sigma & 0 & -\sqrt{6} \\ 0 & 0 & 2\sqrt{3} & 0 & 2 & 0 & 0 \\ 2\sigma & 2\sqrt{3} & -2\sigma^2 & -2\sigma & 2\sqrt{3}\sigma^2 & -2 & 2\sqrt{6}\sigma \\ -1 & 0 & -2\sigma & -3 & 2\sqrt{3}\sigma & 0 & \sqrt{6} \\ -2\sqrt{3}\sigma & 2 & 2\sqrt{3}\sigma^2 & 2\sqrt{3}\sigma & 2\sigma^2 & 2\sqrt{3} & 2\sqrt{2}\sigma \\ 0 & 0 & -2 & 0 & 2\sqrt{3} & 0 & 0 \\ -\sqrt{6} & 0 & 2\sqrt{6}\sigma & \sqrt{6} & 2\sqrt{2}\sigma & 0 & 2 \end{pmatrix} \quad (7.15)$$

is of the form (3.14), with η given by (5.5), and the charge matrix

$$\mathcal{A} = \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \end{pmatrix}, \quad (7.16)$$

which is idempotent of rank two, $\mathcal{A}^3 = 0$, $\mathcal{A}^2 \neq 0$. The question of whether the representative matrices (7.15) and (7.7) can be transformed into each other, or belong to two inequivalent components of the $\text{Tr}(\mathcal{A}^2) = 0$ sector of solution space, remains open.

VIII. APPLICATION TO THE GENERATION OF ROTATING AF SOLUTIONS

Toroidal reduction can also be performed relative to two linearly independent combinations of the three Killing vectors. Replacing e.g. ∂_t by a linear combination of ∂_t and ∂_z simply amounts to changing the values of the parameters M and J , or m and ω . On the other hand, replacing ∂_t by a linear combination of ∂_t and ∂_φ should, as in the four-dimensional Einstein-Maxwell case [14], lead to rotating solutions.

As mentioned in Sect. 4, any solution of EM4 can be lifted to a solution (4.22) of MSG5. Applying this lifting procedure to the four-dimensional electric Bertotti-Robinson solution, with the spacetime geometry $AdS_2 \times S^2$, one obtains [4, 15] a five-dimensional electric Bertotti-Robinson solution with the geometry $AdS_2 \times S^3$, while the four-dimensional magnetic Bertotti-Robinson solution lifts to the five-dimensional magnetic Bertotti-Robinson solution (2.13) with $J = 0$, with the geometry $AdS_3 \times S^2$, and the continuous family of four-dimensional dyonic Bertotti-Robinson solutions lifts to five-dimensional solutions with geometries interpolating between $AdS_2 \times S^3$ and $AdS_3 \times S^2$.

Thus, the EM4 spin-generating mechanism of [14] can be lifted to the case of MSG5 in several fashions. In all cases, this generation will proceed in three steps. First, carry out a transformation Π from an asymptotically flat static solution to the corresponding asymptotically Bertotti-Robinson solution. Second, perform on this the combined transformation

$$d\varphi' = d\varphi - \Omega dt, \quad dt' = \alpha^{-1} dt, \quad (8.1)$$

which does not modify the leading asymptotically Bertotti-Robinson behavior, but modifies the three-dimensional reduced metric $d\sigma^2$. For instance, the reduced metric (3.18) is transformed into

$$d\sigma'^2 = \frac{\hat{\tau}'}{\hat{\tau}} [dr^2 + (r^2 - r_0^2)d\theta^2] + \alpha^2(r^2 - r_0^2)\sin^2\theta d\varphi^2, \quad (8.2)$$

where $\hat{\tau}$ and $\hat{\tau}'$ refer to the untransformed and transformed Bertotti-Robinson metrics, with

$$\hat{\tau}' = \alpha^2 \left[\hat{\tau} - \Omega^2 \hat{\tau}^{-1} \hat{\lambda}_{11} (r^2 - r_0^2) \sin^2\theta \right]. \quad (8.3)$$

Third, transform back with Π^{-1} to an asymptotically flat rotating solution. If the input static solution is uncharged, the output rotating solution will also be uncharged for a suitable value of the parameter α [14].

If the input static solution is a Tangherlini black hole, this procedure should lead [15] to a Myers-Perry black hole. The details have not been spelled out in [15], but to obtain a black hole with two independent angular momenta one should presumably generalize (8.1) to a combined transformation

$$d\varphi' = d\varphi - \Omega_\varphi dt, \quad dz' = dz - \Omega_z dt, \quad dt' = \alpha^{-1} dt. \quad (8.4)$$

The same procedure can be applied to generate a rotating solution from any static solution of EM5 with Tangherlini asymptotics. The application to the (singular) static Emparan-Reall black ring (which has the same asymptotics as a black hole) was carried out in [15] (using for Π the transformation from Tangherlini to the electric Bertotti-Robinson solution, and the transformation (8.1)), with inconclusive results.

The same procedure applied to a static black string, using for Π the transformation from the Schwarzschild black string to the magnetic Bertotti-Robinson solution ((2.13) with $J = 0$) should lead to a rotating black string. Rotating black strings can also be obtained from rotating black holes by the black hole to black string transformation of [12], but it is not clear whether the two procedures always lead precisely to the same solutions. Conceivably, the resulting solutions might differ by higher multipole moments. One could also apply the spin-generating procedure to either a black string or a black hole with a five-dimensional dyonic Bertotti-Robinson solution as intermediate.

This procedure could also in principle be carried out to generate spinning soliton strings or five-dimensional anti-instantons, the magnetic Bertotti-Robinson solution being replaced by the “rotating Bertotti-Robinson” solution equivalent of (5.1) obtained by the coordinate transformation $t \rightarrow z, z \rightarrow -t$ (a $G_{2(+2)}$ transformation).

IX. CONCLUSION

In this paper we have demonstrated the possibility of transforming non-asymptotically flat solutions into asymptotically flat ones using sigma-model maps between different classes of geodesic solutions. This opens a way to construct global black hole solutions starting with near-horizon solutions as seeds. Though we restrained ourselves to the special case of five-dimensional minimal supergravity, this possibility looks general and deserves further study. We have revealed some general features of $AF \leftrightarrow NAF$ maps, and provided a particular realization transforming the Bertotti-Robinson-type solution related to the three-dimensional Gödel black hole into new NUTty or NUTless asymptotically flat soliton ring solutions of MSG5. In the NUTless case, this new ring is horizonless and contains neither conical, nor Misner string singularities. Its physical properties and possible applications await to be investigated.

We have also explored one subtle point in the sigma-model generating techniques concerning transformations between solutions possessing the same reduced three-metric and the same asymptotics, which correspond to geodesics passing through the same point in target space. Such solutions are defined by the tangent vectors to geodesics at this point, so it could be expected that all of them are equivalent under transformations of the isotropy subgroup of the U-duality group. We have shown, however, that in many cases there are obstructions due to the existence of invariants preserved by the isotropy subgroup. As a result, the geodesic solutions generically split into disjoint classes such that the symmetry transformations act only inside each class, but not between different classes. This property does not hold for simple cosets like $SL(2, R)/SO(1, 1)$ or $SU(2, 1)/S[U(2) \times U(1)]$ corresponding to four-dimensional Einstein and Einstein-Maxwell theories respectively, but holds for $SL(2, R)/SO(2, 1)$ (five-dimensional vacuum gravity) and for the coset $G_{2(2)}/((SL(2, R) \times SL(2, R)))$ of MSG5 investigated here, so it presumably is a general feature of large enough cosets. The deeper group-theoretical significance of the above obstructions also awaits to be explored.

Acknowledgments

We wish to thank Paul Sorba for several enlightening discussions on group theory. AB, CMC and DG would like to thank LAPTh Annecy-le-Vieux for hospitality at different stages of this work. DG acknowledges the support of the Russian Foundation of Fundamental Research under the project 14-02-01092-a. The work of CMC was supported by the National Science Council of the R.O.C. under the grant NSC 102-2112-M-008-015-MY3, and in part by the National Center of Theoretical Sciences (NCTS).

Appendix A: $G_{2(+2)}/((SL(2, R) \times SL(2, R)))$ coset representative

The 7×7 matrix M entering Eq. (3.6) was constructed in [18, 19] and has the symmetrical block structure:

$$M = \begin{pmatrix} A & B & \sqrt{2}U \\ B^T & C & \sqrt{2}V \\ \sqrt{2}U^T & \sqrt{2}V^T & S \end{pmatrix}, \quad (\text{A.1})$$

where A and C are symmetrical 3×3 matrices, B is a 3×3 matrix, U and V are 3-component column matrices, and S a scalar. These are given in terms of the moduli by

$$\begin{aligned}
 A &= \begin{pmatrix} [(1-y)\lambda + (2+x)\psi\psi^T - \tau^{-1}\tilde{\omega}\tilde{\omega}^T] & \tau^{-1}\tilde{\omega} \\ +\nu(\psi\psi^T\lambda^{-1}J - J\lambda^{-1}\psi\psi^T) & -\tau^{-1} \\ \tau^{-1}\tilde{\omega}^T & \end{pmatrix}, \\
 B &= \begin{pmatrix} (\psi\psi^T - \nu J)\lambda^{-1} - \tau^{-1}\tilde{\omega}\psi^T J & [(-(1+y)\lambda J - (2+x)\nu + \psi^T\lambda^{-1}\tilde{\omega})\psi \\ + (z - \nu J\lambda^{-1})\tilde{\omega}] \\ \tau^{-1}\psi^T J & -z \end{pmatrix}, \\
 C &= \begin{pmatrix} (1+x)\lambda^{-1} - \lambda^{-1}\psi\psi^T\lambda^{-1} & \lambda^{-1}\tilde{\omega} - J(z - \nu J\lambda^{-1})\psi \\ \tilde{\omega}^T\lambda^{-1} + \psi^T(z + \nu\lambda^{-1}J)J & [\tilde{\omega}^T\lambda^{-1}\tilde{\omega} - 2\nu\psi^T\lambda^{-1}\tilde{\omega} \\ -\tau(1+x-2y-xy+z^2)] \end{pmatrix}, \\
 U &= \begin{pmatrix} (1+x - \nu J\lambda^{-1})\psi - \nu\tau^{-1}\tilde{\omega} \\ \nu\tau^{-1} \end{pmatrix}, \\
 V &= \begin{pmatrix} (\lambda^{-1} + \nu\tau^{-1}J)\psi \\ \psi^T\lambda^{-1}\tilde{\omega} - \nu(1+x-z) \end{pmatrix}, \\
 S &= 1 + 2(x-y),
 \end{aligned} \tag{A.2}$$

with

$$\tilde{\omega} = \omega - \nu\psi, \quad x = \psi^T\lambda^{-1}\psi, \quad y = \tau^{-1}\nu^2, \quad z = y - \tau^{-1}\psi^T J\tilde{\omega}. \tag{A.3}$$

The 7×7 matrix representatives j_M of the infinitesimal generators of $G_{2(+2)}$ may be written in block form

$$j = \begin{pmatrix} S & \tilde{V} & \sqrt{2}U \\ -\tilde{U} & -S^T & \sqrt{2}V \\ \sqrt{2}V^T & \sqrt{2}U^T & 0 \end{pmatrix}, \tag{A.4}$$

where S is a 3×3 matrix, U and V are 3-component column matrices, U^T and V^T the corresponding transposed row matrices, and \tilde{U} , \tilde{V} are the 3×3 dual matrices $\tilde{U}_{ij} = \epsilon_{ijk}U_k$. The matrices $m_a{}^b$, n^a and ℓ_a generating the vacuum $SL(3, R)$ subgroup of $G_{2(+2)}$ are of type S , the corresponding 3×3 blocks being

$$\begin{aligned}
 S_{m_0^0} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_{m_0^1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
 S_{m_1^0} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{m_1^1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
 S_{n^0} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad S_{n^1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\
 S_{\ell_0} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{\ell_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
 \end{aligned} \tag{A.5}$$

The matrices p_a and q are of type U , the corresponding 1×3 blocks being

$$U_{p_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad U_{p_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad U_q = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \tag{A.6}$$

The matrices r^a and t are of type V , the corresponding 1×3 blocks being

$$V_{r^0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad V_{r^1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad V_t = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A.7}$$

Appendix B: Proof that the 3-Gödel solution cannot be transformed to the Schwarzschild black string

The fact that the 3-Gödel solution (5.1) and the Schwarzschild black string (3.22) have the same three-dimensional reduced metric (3.18) suggests that their matrix representatives might be related by a $G_{2(+2)}$ transformation,

$$M_G = P_{SG}^T M_S P_{SG}, \quad (B.1)$$

the corresponding constant matrices η and \mathcal{A} being related by

$$\eta_G = P_{SG}^T \eta_S P_{SG}, \quad \mathcal{A}_G = P_{SG}^{-1} \mathcal{A}_S P_{SG}. \quad (B.2)$$

We prove here that this is impossible.

We first consider the second equation (B.2). The Schwarzschild matrix $\mathcal{A}_S = \text{diag}(1, 0, -1, -1, 0, 1, 0)$ has the three degenerate eigenvalues ± 1 and 0 with the obvious eigenvectors $(\psi_{Si\pm})^a = \delta_{i\pm}^a$ and $(\psi_{Si0})^a = \delta_{i0}^a$. The matrix \mathcal{A}_G has the same degenerate eigenvalues ± 1 and 0 with suitably orthonormalized eigenvectors $(\psi_{Gi\pm})^a$ and $(\psi_{Gi0})^a$. The similarity transformation, given by the sum $P_{SG} = \psi_{Sk\alpha} \psi_{Gk\alpha}^T$, is thus

$$(P_{SG})^a_b = (\psi_{Ga})^b. \quad (B.3)$$

Now let us compute, from the first equation (B.2),

$$(\eta_G)_{77} = (\eta_S)_{ab} (\psi_{Ga})^7 (\psi_{Gb})^7 = [(\psi_{Gi0})^7]^2 - [(\psi_{Gi+})^7]^2 - [(\psi_{Gi-})^7]^2 \quad (B.4)$$

(with sum over repeated indices implied). Remembering that ψ_{Gi0} solves $\mathcal{A}_G \psi_{Gi0} = 0$, we find from the second and fourth line of (5.6) that $(\psi_{Gi0})^7 = 0$, leading to $(\eta_G)_{77} < 0$, in contradiction with (5.5).

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